

MAS231 GEOMETRY II: KNOTS AND SURFACES

25-Apr-05 14:30

Duration 2 h 0 min

**YOU ARE NOT PERMITTED TO START READING THIS
QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN
INVIGILATOR**

Queen Mary, University of London

B.Sc. Examination by Course Units 2005

MAS 231 GEOMETRY II: KNOTS & SURFACES

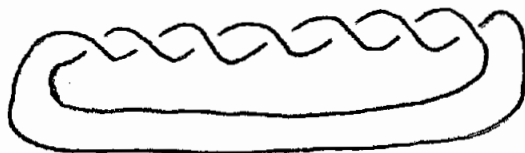
Date and time: 25th April, 2005 at 14:30

Duration: 2 hours

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted. Electronic calculators may not be used. The unauthorized use of a calculator constitutes an examination offense.

1. (a) Describe the three types of Reidemeister moves on knot diagrams and state the theorem of Reidemeister.

(b) State what it means for a knot diagram to be tricolourable and for tricolourability to be a knot invariant. Show that the following knot with seven crossings is *not* tricolourable:



(c) Consider the same type of knot as in part (b) but with an odd number n crossings in place of seven. For which n is the knot tricolourable? You are *not* asked to write out a formal proof.

(d) Define the writhe W of a knot diagram and give an example to demonstrate that W is *not* a true knot invariant.

(e) Show that W is invariant under a move of type II and state which of the other types of move it is invariant under.

Please turn over

2. (a) Give an inductive definition of the Kauffman bracket B of a knot or link diagram and show that

$$B(\boxed{\text{diagram}}) = (-x)^3 B(\text{diagram})$$

where the box denotes the remainder of the knot or link.

- (b) Give an example to demonstrate that B is *not* a true knot invariant.

- (c) Define the Jones knot invariant $J(t)$ in terms of B and the writhe W , and compute it for the knot diagram in Question 1(b) (only the diagram is needed from that question). You may assume that $W = 7$ for this knot diagram, and that

$$B(\text{six crossings}) = -x^{16} + x^{12} - x^8 + x^4 - 1 - x^{-8}$$

- (d) Consider the same type of knot as in Question 1(b) but with an odd number n crossings in place of seven. The general formula for its Jones invariant is then

$$J(t) = t^{\frac{n-1}{2}} \left(1 + t^2 \left(\frac{1 - t^{n-1}}{1 + t} \right) \right).$$

Verify this assertion for $n = 1$.

3. (a) Define what it means for a curve to be regular. Sketch the curve

$$\gamma(t) = (\cos^3 t, \sin^3 t, 0), \quad t \in (0, 2\pi)$$

and say if it is regular. Find and mark the directions of the unit tangent vector $\tau = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}$ at just after $t = 0$ and just before $t = \frac{\pi}{2}$.

- (b) Define the signed curvature K_S of a regular planar curve γ in the $x - y$ -plane and compute it for the curve in part (a) restricted to $t \in (0, \frac{\pi}{2})$.

- (c) Let γ be a regular curve with unit tangent vector τ and let $\mathbf{n}$ be the vector τ rotated by $\pi/2$ anticlockwise. Assuming the equation

$$\dot{\tau} = \mathbf{n} K_S \|\dot{\gamma}\|,$$

show that $K_S \|\dot{\gamma}\| = \dot{\theta}$, where θ is the anticlockwise angle of τ from the x -axis.

- (d) With reference to your sketch in part (a), verify that the total anticlockwise angular change of direction for this curve restricted to $t \in (0, \frac{\pi}{2})$ agrees with $\int_0^{\frac{\pi}{2}} K_S \|\dot{\gamma}\| dt$, where K_S is from part (b).

4. (a) Define the curvature K and torsion T of a curve in $\mathbb{R}^3$. State a theorem about the extent to which a unit-speed curve in $\mathbb{R}^3$ is determined by its curvature and torsion.

(b) Show that if γ is a unit-speed curve then $\dot{\gamma}$ and $\ddot{\gamma}$ are orthogonal. Hence, or otherwise, show that the curvature in this case is $K = \|\ddot{\gamma}\|$.

(c) Show that a curve with non-vanishing curvature has zero torsion if and only if it is planar. You may assume that the curve is unit speed, and the equation $\dot{\beta} = -T\nu$, where ν, β are the principal normal and conormal respectively.

(d) Show that the curve

$$\gamma(t) = \frac{1}{\sqrt{2}}(t \cos(\ln(t)), t \sin(\ln(t)), 0), \quad t \in (0, \infty)$$

is unit speed and has curvature $K = 1/t$. What is its torsion? You may use part (c).

5. (a) Sketch the surface with single surface patch

$$\sigma(u, v) = (u, v, e^{-(u^2+v^2)}), \quad U = \mathbb{R}^2$$

(b) Find the standard unit normal N_σ of the surface in part (a) and mark it at a typical point on your sketch.

(c) Find the *first* fundamental form F_I of the surface in (a) and obtain an integral expression for the area of the part of the surface given by a region $R \subset U$.

(d) Find the *second* fundamental form F_{II} of the surface in (a).

(e) State what it means in terms of the Gauss curvature K_G for a point to be (i) elliptic (ii) hyperbolic (iii) parabolic or locally flat. Using (d), find and show on your sketch which parts of the surface in (a) are of each type.

6. (a) Let S be a surface of revolution with surface patch of the form

$$\sigma(u, v) = (f(u) \cos(v), f(u) \sin(v), g(u)), \quad f > 0, \quad \left(\frac{df}{du}\right)^2 + \left(\frac{dg}{du}\right)^2 = 1$$

and first fundamental form $du^2 + f(u)^2 dv^2$. Show that the second part

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

of the geodesic equations implies that $\Omega = f^2\dot{v}$ is constant. Also show that $\dot{u} = \pm\sqrt{1 - \frac{\Omega^2}{f^2}}$ (you may assume that the geodesic is unit speed).

Please turn over

(b) Let $f(u) = \cos(u)$, $g(u) = \sin(u)$ for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that the surface in part (a) is a unit sphere. Find the values of Ω for the following geodesics: (i) the equator, (ii) the curve of constant $v = \frac{\pi}{2}$ and (iii) the curve of constant $v = 0$. Show these geodesics on a sketch.

(c) State the Gauss-Bonnet theorem for curvilinear polygons. Hence, on a unit sphere, prove that the area contained between the three geodesics in part (c), taken in order in the upper hemisphere, is $\frac{\pi}{2}$. You may assume that $K_G = 1$ and that these geodesics intersect at right angles.

End of Exam Paper